Using Leslie matrices as the application of eigenvalues and eigenvectors in a first course in Linear Algebra

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Linear Algebra and its Applications

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Ch 1 Linear Equations in Linear Algebra (9)
Markov matrices and page ranking algorithms.

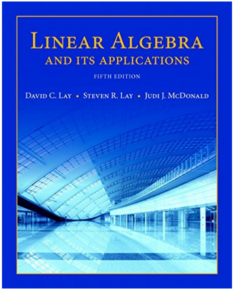
Ch 2 Matrix Algebra (3)

Ch 3 Determinants (3)

Ch 4 Vector Spaces (6)
The Lagrange and Newton bases.

Ch 5 Eigenvalues and Eigenvectors (6)
The Leslie age distribution model.

Ch 6 Orthogonality and Least Squares (6)
Least-Squares Problems



Why the Leslie matrix?

Talk Outline

1. The Leslie population growth model.
2. It's a linear transformation!
3. The dominant eigenvalue and eigenvector.
4. Questions we can ask students.
5. Resources in the the paper.

The Leslie population growth model.

Divide the females in a population into age groups G_1, G_2, ..., G_n.
Model fertility rates f_k and survival probabilities s_k.

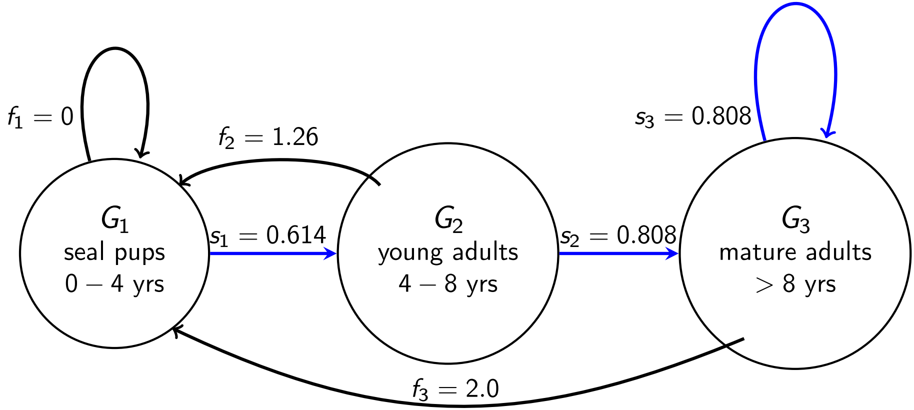


Figure: Leslie model for grey seal population on Sable Is.

It's a linear transformation!

Let p_i^t be the number of females in G_i at time t.
Let $P^{(t)} = [p_1^t, p_2^t, \ldots, p_n^t]$ be the population vector at time t. Then

$$P^{(t+1)} = \begin{bmatrix} f_1 p_1^t + f_2 p_2^t + f_3 p_3^t \\ s_1 p_1^t \\ s_2 p_2^t + s_3 p_3^t \end{bmatrix}$$

$$P^{(t+1)} = \underbrace{\begin{bmatrix} f_1 & f_2 & f_3 \\ s_1 & 0 & 0 \\ 0 & s_2 & s_3 \end{bmatrix}}_{\text{Leslie Matrix } L} \begin{bmatrix} p_1^t \\ p_2^t \\ p_3^t \end{bmatrix}$$

```
> L := Matrix([[0.0,1.26,2.00],[0.624,0,0],[0,0.808,0.808]]);
```

$$L := \begin{bmatrix} 0 & 1.26 & 2.0 \\ 0.614 & 0 & 0 \\ 0 & 0.808 & 0.808 \end{bmatrix}$$

```
> P[0] := <1.0,1.0,1.0>:
> for i to 16 do P[i] := L.P[i-1]; od:
> P[0],P[1],P[15],P[16];
```

$$\begin{bmatrix} 1.0 \\ 1.0 \\ 1.0 \end{bmatrix}, \begin{bmatrix} 3.260 \\ 0.624 \\ 1.616 \end{bmatrix}, \begin{bmatrix} 798.283951 \\ 332.370572 \\ 388.807105 \end{bmatrix}, \begin{bmatrix} 1196.40113 \\ 498.129186 \\ 582.711563 \end{bmatrix}$$

The grey seal population has exploded!

Let $D^{(t)}$ be the population distribution vector at time t.
So $D^{(t)} = P^{(t)} / \sum_{i=1}^{n} P_i^{(t)}$.

```
> local D: # by default D is the differential operator in Maple
> pop := proc(v) local i; add(v[i],i=1..numelems(v)) end:
> for i from 0 to t do D[i] := P[i]/pop(P[i]); od:
> D[0],D[1],D[15],D[16];
```

$$\begin{bmatrix} 0.333333333 \\ 0.333333333 \\ 0.333333333 \end{bmatrix}, \begin{bmatrix} 0.592727273 \\ 0.113454545 \\ 0.293818182 \end{bmatrix}, \begin{bmatrix} 0.525372893 \\ 0.218742327 \\ 0.255884780 \end{bmatrix}, \begin{bmatrix} 0.525372883 \\ 0.218742326 \\ 0.255884791 \end{bmatrix}$$

Let $D^{(t)}$ be the population distribution vector at time t.
So $D^{(t)} = P^{(t)} / \sum_{i=1}^{n} P_i^{(t)}$.

```
> local D: # by default D is the differential operator in Maple
> pop := proc(v) local i; add(v[i],i=1..numelems(v)) end:
> for i from 0 to t do D[i] := P[i]/pop(P[i]); od:
> D[0],D[1],D[15],D[16];
```

$$\begin{bmatrix} 0.333333333 \\ 0.333333333 \\ 0.333333333 \end{bmatrix}, \begin{bmatrix} 0.592727273 \\ 0.113454545 \\ 0.293818182 \end{bmatrix}, \begin{bmatrix} 0.525372893 \\ 0.218742327 \\ 0.255884780 \end{bmatrix}, \begin{bmatrix} 0.525372883 \\ 0.218742326 \\ 0.255884791 \end{bmatrix}$$

Thus $D^{(t)}$ has converged to an eigenvector of L with eigenvalue

```
> seq( P[16][i]/P[15][i], i=1..3 );
```

$$1.49871625062797, \; 1.49871627642614, \; 1.49871634352741$$

The Dominant Eigenvalue and Eigenvector

Theorem

For any non-zero initial population $P^0 = [p_1^0, p_1^0, \ldots, p_n^0]$, if at least one fertility rate f_i is positive, the Leslie matrix L has a unique positive eigenvalue λ^+. If v^+ is the corresponding eigenvector and at least two consecutive fertility rates are positive, λ^+ is dominant and the population distribution will converge to an eigenvector of L, that is $\lim_{t\to\infty} D^{(t)}$ exists and is a multiple of v^+.

We also have the following physical interpretation for λ^+.

$\lambda^+ < 1$ means the population will decline exponentially.
$\lambda^+ > 1$ means the population will grow exponentially.
$\lambda^+ = 1$ means the population is stable, it does not change.

Grey Seals and Northern Spotted Owls

Grey Seals

Age	0–4yr	4–8yrs	$\geq$ 8yrs
f_i	0	1.26	2.00
s_i	0.604	0.808	0.808

Leslie matrix

$$\begin{bmatrix} 0 & 1.26 & 2.00 \\ 0.614 & 0 & 0 \\ 0 & 0.808 & 0.808 \end{bmatrix}$$

Figure: Sable island grey seal data and Leslie matrix: $\lambda^+ = 1.50$

Spotted Owls

Age	0–1yr	1–2yrs	$\geq$ 2yrs
f_i	0	0	0.33
s_i	0.18	0.71	0.94

Leslie matrix

$$\begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix}$$

Figure: Northern spotted owl data and Leslie matrix: $\lambda^+ = 0.91$

To compute λ^+ one must solve a cubic polynomial, easy with Maple.

Population Growth Control

```
> L := Matrix([[0.0,1.26,2.00],[s1,0,0],[0,0.808,0.808]]);
```

$$L := \begin{bmatrix} 0 & 1.26 & 2.0 \\ s1 & 0 & 0 \\ 0 & 0.808 & 0.808 \end{bmatrix}$$

To stabilize the population: what must s_1 be so that $\lambda^+ = 1$?

$$Lx = 1x \implies (L - I)x = 0 \implies \det(L - I) = 0.$$

Population Growth Control

```
> L := Matrix([[0.0,1.26,2.00],[s1,0,0],[0,0.808,0.808]]);
```

$$L := \begin{bmatrix} 0 & 1.26 & 2.0 \\ s1 & 0 & 0 \\ 0 & 0.808 & 0.808 \end{bmatrix}$$

To stabilize the population: what must s_1 be so that $\lambda^+ = 1$?

$$Lx = 1x \implies (L - I)x = 0 \implies \det(L - I) = 0.$$

```
> I3 := IdentityMatrix(3):
> Determinant(L-I3) = 0;
```

$$-0.192 + 1.85792\, s1 = 0$$

```
> s1 = solve(Determinant(L-I3) = 0);
```

$$s1 = 0.1033413710$$

Population Growth Control

```
> L := Matrix([[0.0,1.26*f,2.00*f],[0.694,0,0],[0,0.808,0.808]]);
```

$$\begin{bmatrix} 0 & 1.26\,f & 2\,f \\ 0.604 & 0 & 0 \\ 0 & 0.808 & 0.808 \end{bmatrix}$$

What must f be so that $\lambda^{+} = 1$?

```
> I3 := IdentityMatrix(3):
> Determinant(L-I3) = 0;
```

$$-0.192 + 1.12218368 f = 0$$

```
> f = solve(Determinant(L-I3) = 0);
```

$$f = 0.1710949851$$

Some info on Leslie Matrices

A Leslie matrix is an n by n matrix of the form

$$L = \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \\ s_1 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & s_{n-1} & \mathbf{0} \end{bmatrix}$$

where $n \geq 2$, the survival rates $s_i > 0$ and fertility rates $f_i \geq 0$ with at least one $f_i > 0$.

Patrick H. Leslie
The use of matrices in certain population mathematics.
Biometrika, **33**(3), 183–212, 1945.

Resources in the paper.

- Exercises with nice matrices e.g. $L = \begin{bmatrix} 0 & \frac{7}{6} & \frac{7}{6} \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$ Here $\det(A - xI) = -x^3 + \frac{2}{3}x^2 + \frac{7}{12}x$ so L has eigenvalues $0, \frac{7}{6}, -\frac{1}{2}$.
- Some exercises with the eigenvalues and eigenvectors.
- Some some population control exercises.
- An Appendix of real data.
  Data is for Canadian female population in 1965 from Anton.
  Data for a New Zealand sheep population from Anton.
  Data for North Amerian woodland caribou from Poole.
  The Sable Island grey seal data from Manske, Schwarz and Stobo.

Thank you for attending!