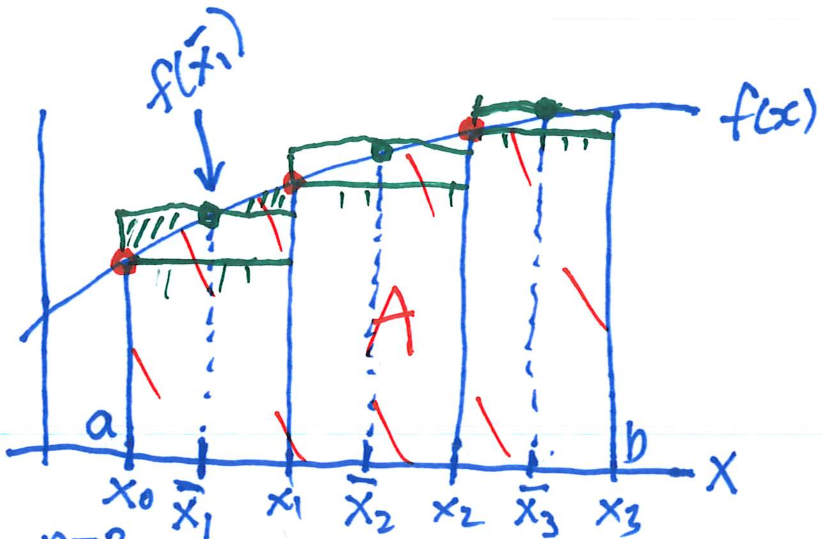
5.2 The Definite Integral

Assignment #1 due 11pm Tuesday Sept. 16th.

The Midpoint Rule M_n



$n=3$
$n=6$

Divide $[a,b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\dots$ $[x_{n-1}, x_n]$ of equal width $\Delta x = (b-a)/n$.

Let $\bar{x}_i$ be the midpoint of $[x_{i-1}, x_i]$. So
$\bar{x}_i = (x_{i-1} + x_i)/2.$

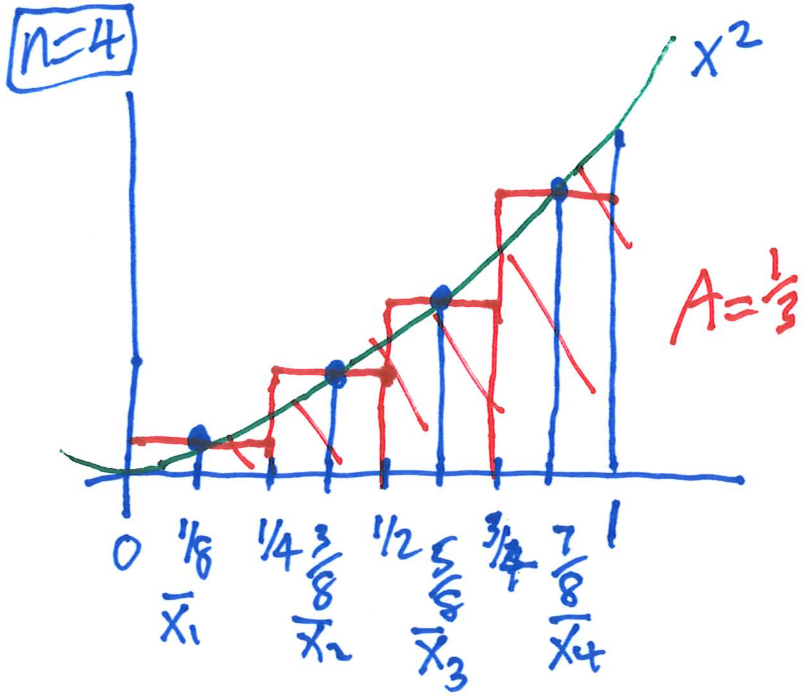
Approximate A with n rectangles

$$M_n = \Delta x\, f(\bar{x}_1) + \Delta x\, f(\bar{x}_2) + \dots + \Delta x\, f(\bar{x}_n)$$

Theorem: $\lim_{n \to \infty} M_n = A.$

Let's test this theorem on an example.

Example $f(x)=x^2$, $a=0$, $b=1$



$$M_4 = \frac{1}{4}\cdot f\left(\tfrac{1}{8}\right) + \frac{1}{4} f\left(\tfrac{3}{8}\right) + \frac{1}{4} f\left(\tfrac{5}{8}\right) + \frac{1}{4} f\left(\tfrac{7}{8}\right)$$

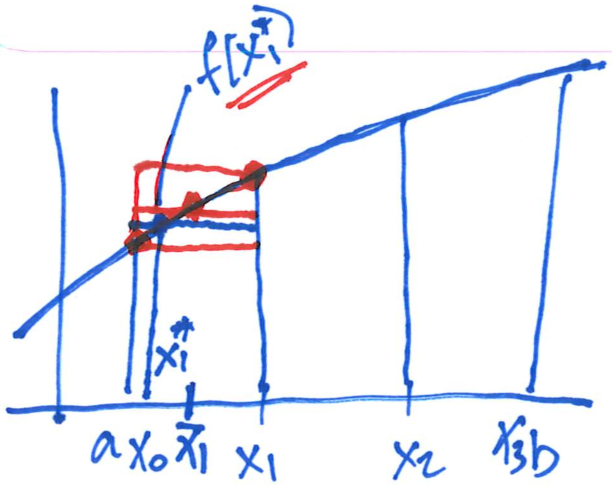
$$= \frac{1}{4}\left(\frac{1}{64} + \frac{9}{64} + \frac{25}{64} + \frac{49}{64}\right)$$

$$= 84/256 = .\underline{32}8125$$

n	L_n	R_n	M_n
4	.21875	.46875	.$\underline{3}$28125
1000	.$\underline{33}$283	.$\underline{33}$383	.$\underline{333333}$23
10^6			

M_n is a much better approx. of A than L_n and R_n.

Riemann



If you pick x_i^* on $[x_{i-1}, x_i]$, i.e., $x_{i-1} \le x_i^* \le x_i$ and we appoximate A with n rectangles

$$S_n = \sum_{i=1}^{n} \underset{\substack{\uparrow \\ \text{width}}}{\Delta x}\ \underset{\substack{\uparrow \\ \text{height.}}}{f(x_i^*)}$$

then $\boxed{\lim_{n\to\infty} S_n = A.}$ ← Riemann Sum.

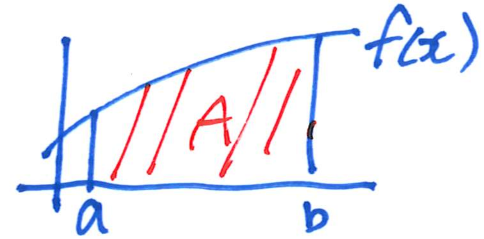
If $x_i^* = \bar{x}_i$ then $S_n = M_n$.

Leibniz.

Let $f(x)$ be continuous on $[a,b]$. Define the <u>definite integral</u> by

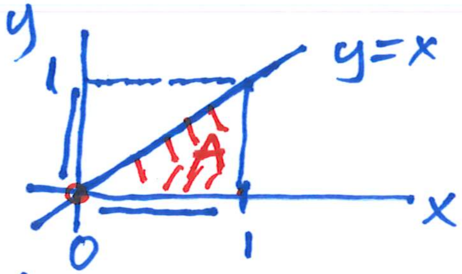
integral symbol → ; b ← upper limit

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \cdot f(x_i^*) =$$

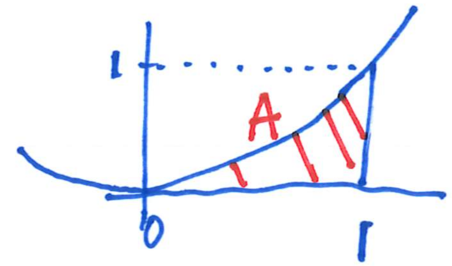


↑ lower limit ↑ integrand ??

What is $\int_0^1 x \, dx = \frac{1}{2}$

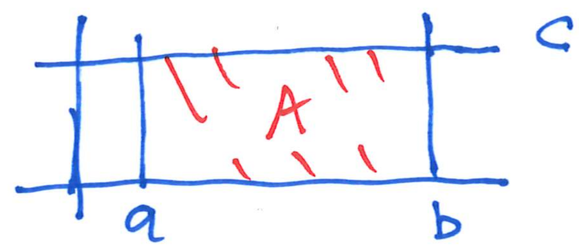


What is $\int_0^1 \underline{x^2} \, dx = \frac{1}{3}$



What is $\int_a^b c \, dx = \underset{\substack{\uparrow \\ \text{height}}}{c} \cdot \underset{\substack{\uparrow \\ \text{width}}}{(b-a)}$

$\boxed{a \le b}$

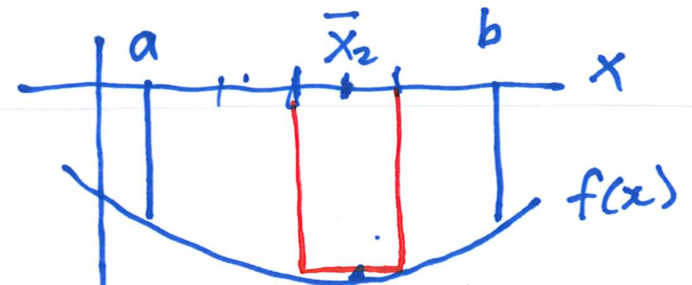


Properties of Definite Integrals $\boxed{b \geq a}$

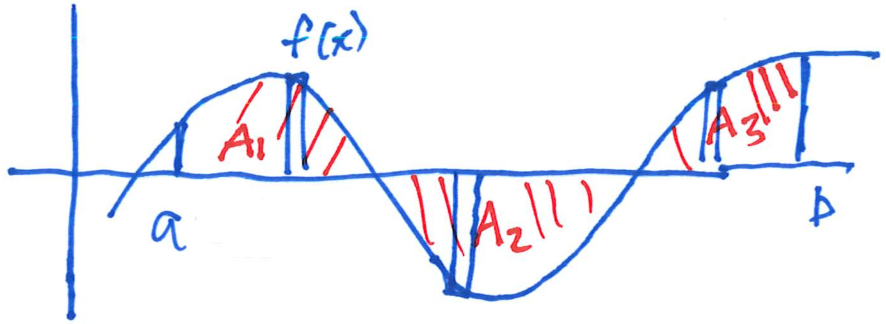
① If $f(x) \geq 0$ on $[a,b]$ then $\int_a^b f(x)\underline{dx} \geq 0$.

② If $f(x) \leq 0$ on $[a,b]$ then $\int_a^b f(x)\,dx \leq 0$.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underset{>0}{\Delta x} \cdot \underset{<0}{f(x_i^*)} \leq 0$$



$\underset{>0}{\Delta x} \cdot \underset{<0}{f(\bar{x}_2)} < 0.$

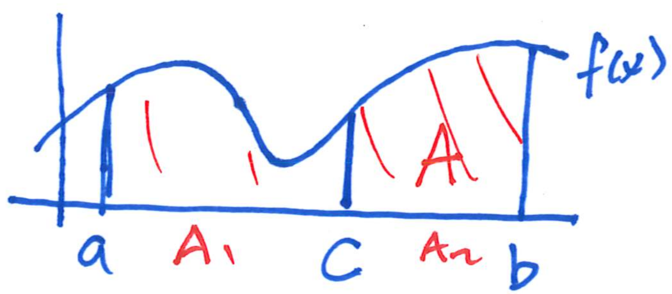


$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$$

③ $$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$ [Sum Rule]

④ $$\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$$

Proof. $$\int_a^b c\cdot f(x)\,dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n} \Delta x\, c\cdot f(x_i^*)\right) = c\Big(\underbrace{\lim_{n\to\infty}\sum_{i=1}^{n}\Delta x\, f(x_i^*)}_{\int_a^b f(x)\,dx}\Big)$$

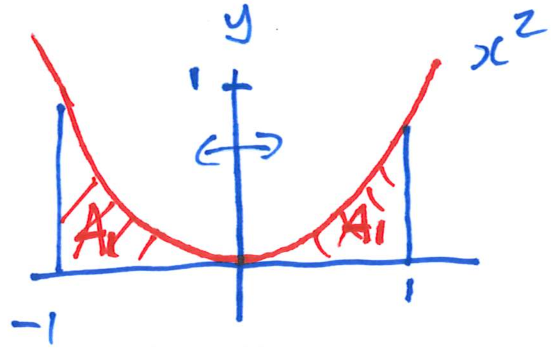
⑤ 

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$
$$A = A_1 + A_2$$

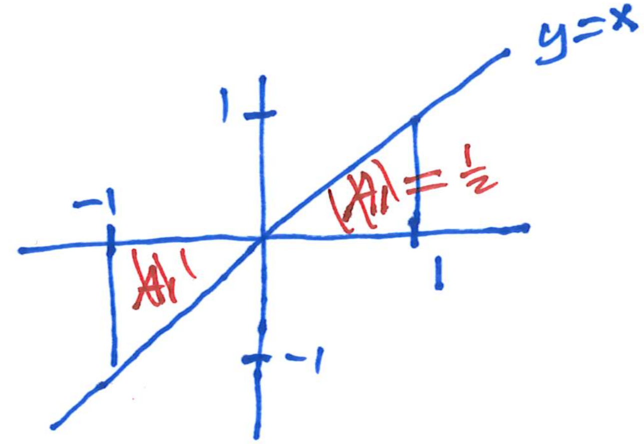
Ex 1. Calculate $\int_0^1 (\underset{\substack{\uparrow \\ f(x)}}{2x^2} + \underset{\substack{\uparrow \\ g(x)}}{x})\,dx \overset{(P3)}{=} \int_0^1 2x^2\,dx + \int_0^1 x\,dx$

$\overset{(P4)}{=} 2\cdot\int_0^1 x^2\,dx + \int_0^1 x\,dx = 2\cdot\frac{1}{3} + \frac{1}{2} = \frac{2}{3} + \frac{1}{2} = \frac{7}{6}.$

Ex 2. Calculate $\int_{-1}^1 x^2\,dx \overset{(P5)}{=} \underbrace{\int_{-1}^0 x^2\,dx}_{= 1/3 \text{ by symmetry.}} + \underbrace{\int_0^1 x^2\,dx}_{= 1/3 \text{ last day}} = \frac{2}{3}$



Ex 3. Calculate $\int_{-1}^1 x\,dx = -A_1 + A_1 = \underline{0}$



The Fundamental Theorem of Calculus! Friday