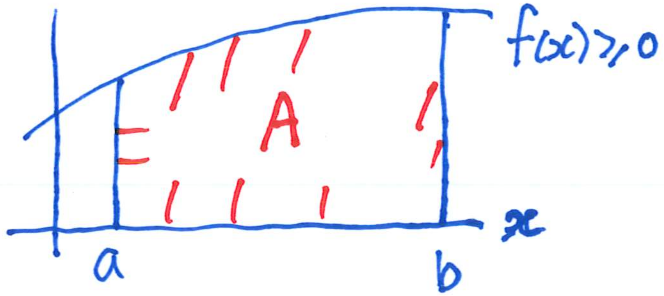
5.3 The Fundamental Theorem of Calculus



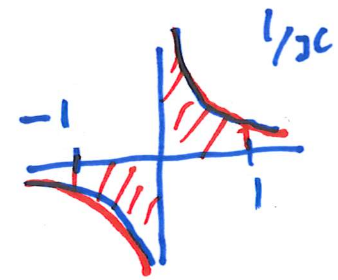
Riemann Sum

$$= \lim_{n\to\infty} \boxed{\sum_{i=1}^{n} \underbrace{\Delta x}_{\text{width}} \, \underbrace{f(x_i^*)}_{\text{height}}} = \int_a^b f(x)\,dx.$$

How can we calculate A?

The Fundamental Theorem of Calculus (FTC)

Let f(x) is continuous on $[a,b]$.



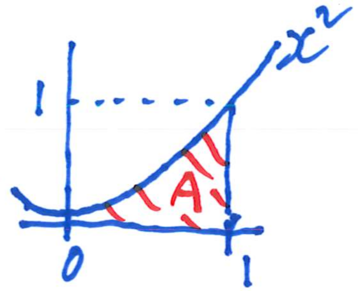
is not continuous on $[-1,1]$

Part (1) If $g(x) = \int_a^{x} f(t)\,dt$ then $g'(x) = f(x)$

Part (2) If $F'(x) = f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$.

F(x) is an antiderivative of f(x)

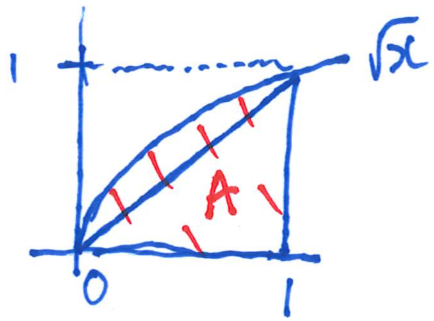
Ex 1.



$$A = \int_0^1 x^2\,dx = F(1) - F(0) = \tfrac{1}{3}\cdot 1^3 - \tfrac{1}{3}\cdot 0^3 = \tfrac{1}{3} \quad ✓$$

$$F(x) = \tfrac{1}{3}(x^3) \qquad F'(x) = \tfrac{1}{3}(3x^2) = x^2 \quad ✓$$

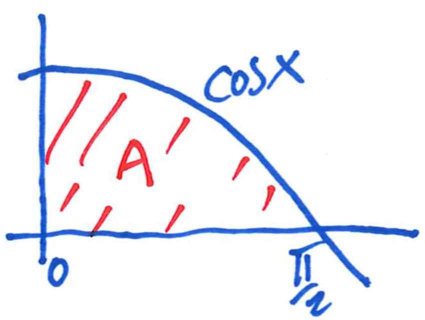
Ex 2.



$$A = \int_0^1 \sqrt{x}\,dx = F(1) - F(0) = \tfrac{2}{3}\cdot 1^{3/2} - \tfrac{2}{3}\cdot 0^{3/2} = \tfrac{2}{3}.$$

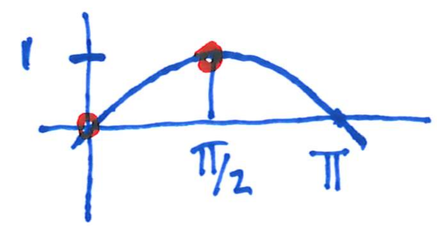
$$f(x) = \sqrt{x} = x^{1/2} \qquad F(x) = \tfrac{2}{3}x^{3/2}$$

Ex 3

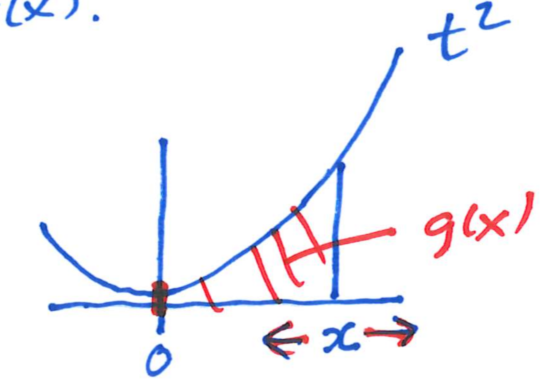


$$A = \int_0^{\pi/2} \cos x\,dx = \sin\tfrac{\pi}{2} - \sin 0 = 1 - 0 = 1.$$

$$F(x) = \sin x$$



FTC (1) If $g(x) = \int_a^x f(t)\,dt$ then $g'(x) = f(x)$.



Ex 4

Find $g(x) = \int_0^{x} \underset{\uparrow f(t)}{t^2}\,dt$.

By FTC(2) $g'(x) = f(x) = x^2$

$\Rightarrow \quad g(x) = \frac{1}{3}x^3 + C$

$g(0) = 0 + C = 0 \Rightarrow C = 0$

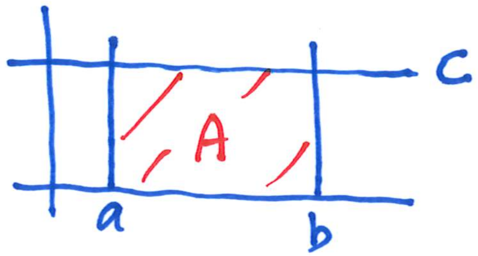
$g(0) = \int_0^0 t^2\,dt = 0$

$\Rightarrow \quad g(x) = \frac{1}{3}x^3$.

Notation. Define $[F(x)]_a^b = F(b) - F(a)$.

Ex 5. $\int_0^1 \underset{\uparrow f(x)}{x^2}\,dx = \underset{F(x)}{\left[\frac{1}{3}x^3\right]_0^1} = \underset{F(1) - F(0)}{\frac{1}{3}\cdot 1^3 - \frac{1}{3}\cdot 0^3} = \frac{1}{3}$

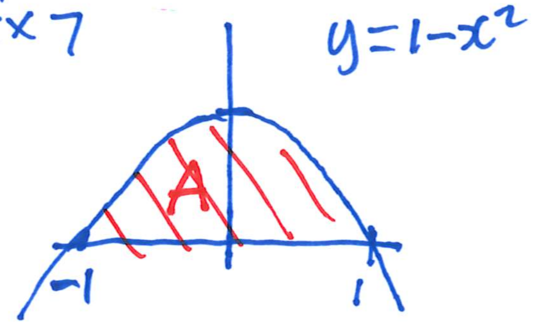
Ex 6



$A = (b-a)\cdot c$

$$A = \int_a^b \underset{f(x)}{\underset{\uparrow}{c}}\, dx = \Big[\, \underset{F(x)}{\underset{\uparrow}{c\cdot x}} \,\Big]_a^b = \underbrace{c\cdot b - c\cdot a}_{F(b)-F(a)} = c(b-a).$$

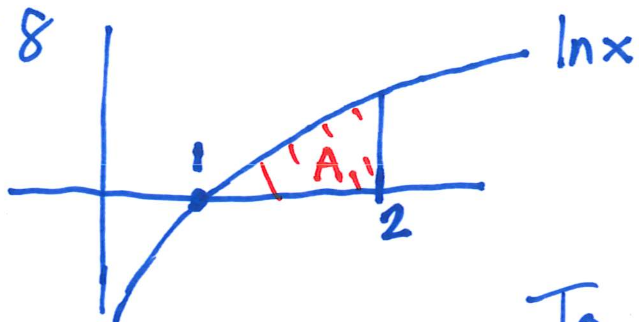
Ex 7



$$A = \int_{-1}^{1} (1-x^2)\cdot dx = \Big[\, x - \tfrac{1}{3}x^3 \,\Big]_{-1}^{1} = \underbrace{\left(1 - \tfrac{1}{3}\right)}_{F(1)} - \underbrace{\Big(-1 + \overset{\overset{-2/3}{/\!/}}{\tfrac{1}{3}(+1)}\Big)}_{F(-1)}$$

$$= 2/3 + 2/3 = 4/3.$$

Ex 8



$$A = \int_1^2 \underline{\ln x}\, dx = \Big[\quad ? \quad\Big]_1^2$$

To use the FTC we need an antiderivative.

How difficult is it to find antiderivatives?

The FTC   Let $f(x)$ be continuous on $[a,b]$.

FTC (1)   If $g(x)=\int_a^x f(t)\,dt$ then $\boxed{g'(x)=f(x)}$

FTC (2)   If $\boxed{F'(x)=f(x)}$ then $\int_a^b f(x)\,dx = F(b)-F(a)$.

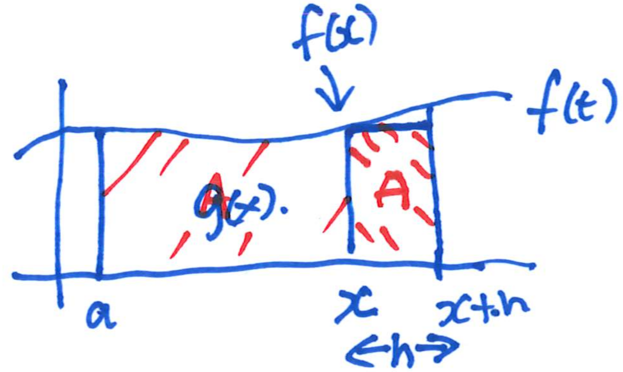
Proof (1) $\Rightarrow$ (2) $F(x)$ and $g(x)$ are both antiderivatives of $f(x)$.

✓ By Th 1 of 4.9. $F(x) \overset{x=b}{=} g(x)+C$.

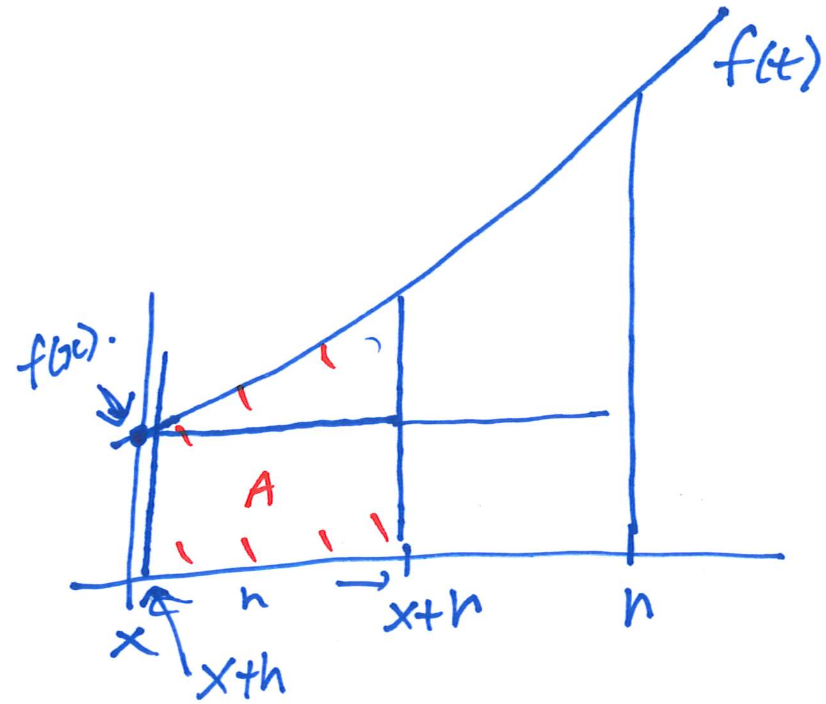
$$\begin{aligned}
F(b)-F(a) &= (g(b)+C)-(g(a)+C)\\
&= g(b)-g(a)\\
&= \int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{0}\\
&= \int_a^b f(x)\,dx.
\end{aligned}$$

Proof of (1).

$$g(x) = \int_a^x f(t)\,dt$$



$$g'(x) = f(x)$$



Consider $A = g(x+h) - g(x) \approx h\cdot f(x)$.

$$\Rightarrow \quad \frac{g(x+h)-g(x)}{h \to 0} \overset{=}{\approx} f(x)$$

Claim. $$\Rightarrow \quad \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \overset{=}{\approx} f(x)$$

$$\Rightarrow \quad \overset{\shortparallel}{g'(x)} = f(x).$$