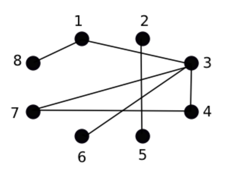
Crossings and nestings in four combinatorial objects

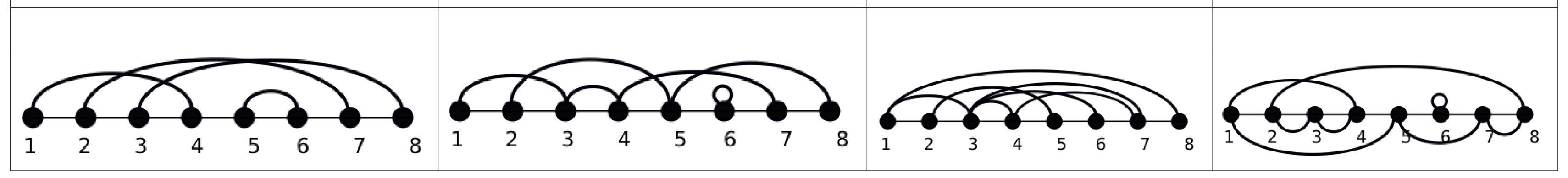
Sophie Burrill
Simon Fraser University *Thank you to NSERC for support*

Perspective

The remarkable result of equidistribution was proven for matchings and partitions by Chen et al. in 2005. Since then, there has been a great deal of work done on crossing and nesting in other combinatorial objects, specifically permutations and graphs. Our goal is to understand the distribution of these four structures according to crossings and nestings using bijections, generating functions and continued fractions. We also give a new result and make a conjecture about the equidistribution of permutations.

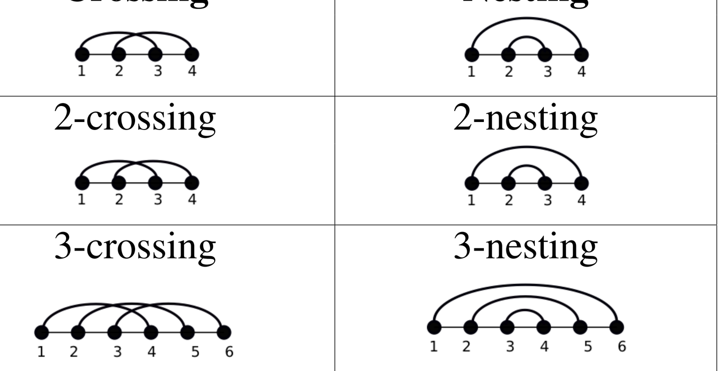
Four combinatorial families

Matchings	Partitions	Graphs	Permutations
$\{\{1,4\},\{2,7\},\{5,6\},\{3,8\}\}$	1 3 4 7 | 2 5 8 | 6		$\begin{bmatrix} 1&2&3&4&5&6&7&8 \\ 4&8&2&3&1&6&5&7 \end{bmatrix}$



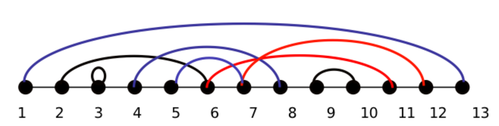
Crossings and Nestings

Crossing	Nesting
2-crossing	2-nesting
3-crossing	3-nesting
k-crossing k mutually crossing arcs	k-nesting k mutually nesting arcs



Definition An object is k-noncrossing (k-nonnesting) if none of the arcs form a k-crossing (k-nesting).

Example.



This is an example of a 4-noncrossing and 4-nonnesting partition. A crossing (2-crossing) is illustrated above in red. A 3-nesting is illustrated in blue.

Equidistribution

Define:

$Ne(k,n) :=$	the total number of k-nestings in an object of size n;
$Cr(k.n) :=$	the total number of k-crossing in an object of size n.
$F(i,j,n) :=$	the total number of objects with i crossings and j nestings.
$N(n,i,j) :=$	the number of objects of size n with maximum nesting of size i, crossing of size j.

A summary of results:

Object	$Ne(1,n)=Cr(1,n)$	$Ne(2,n)=Cr(2,n)$	$F(i,j,n)=F(j,i,n)$	$N(n,i,j)=N(n,j,i)$
Matching	folklore	deS83	Ch07	Ch07
Partitions	folklore	FoZe90	Ch07	Ch07
Graphs			deM07	
Permutations		FoZe90	Co07	Conj.

Revised statistics

Notice that a singleton under an arc is not considered a nesting for a partition, nor is two arcs that connect at a single vertex considered a crossing. Despite this, alternative definitions for crossings and nestings can be defined such that this is the case, specifically *enhanced* crossings and nestings.

References

FoZe90: D. Foata, D. Zeilberger *Denert's permutation statistic is indeed Euler-Mahonian* Stud. Appl. Math. **83** (1) 31-59, 1990.

Ch07: W.Y.C. Chen, E.Y.P Dend, R.R.X Du, R.P. Stanley, and C.H. Yan, *Crossings and nestings of matchings and partitions*, Trans. Amer. Math. Soc. **359**, 1555-1575, 2007.

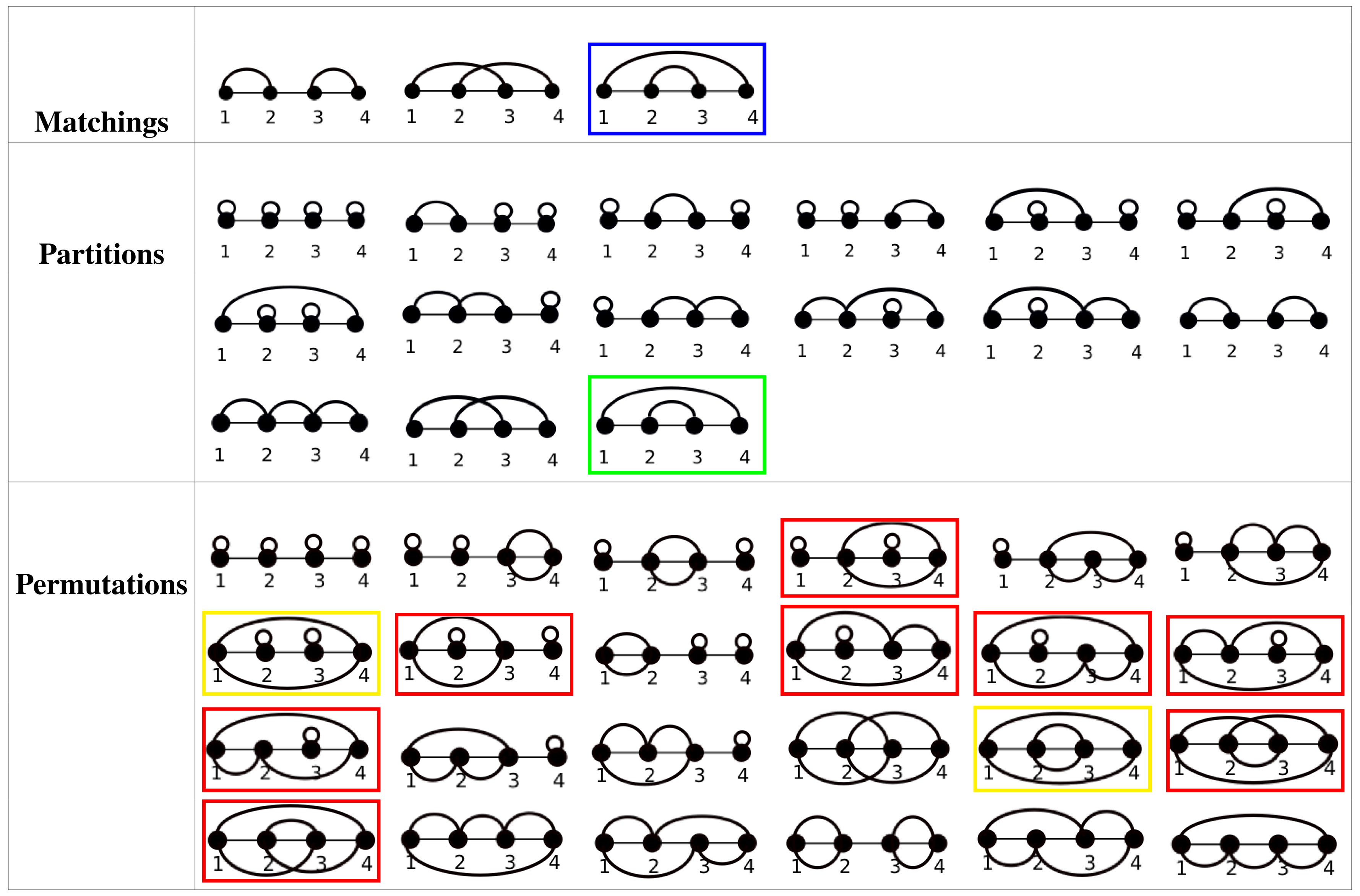
Co07: S. Corteel, *Crossings and alignments of permutations*, Adv. in App. Math. **38**, 149-163. 2007.

deM07: A. de Mier, *k-noncrossing and k-nonnesting graphs and fillings of Ferrers diagrams*, Combinatorica **27**, no. 6, 699–720, 2007.

deS83: M. de Sainte-Catherine, *Couplages et Pfaffiens en combinatoire, physique et informatique*, Ph.D. Thesis, University of Bordeaux I, 1983.

Arc annotated sequences for $n = 4$

In the following table we show all arc annotated sequences for matchings, partitions and permutations when $n = 4$. We place similarly colored boxes around sequences which have the same number of 2-nestings.



Bijections with lattice paths

Matchings

Bijection Φ_1 (Fl80)

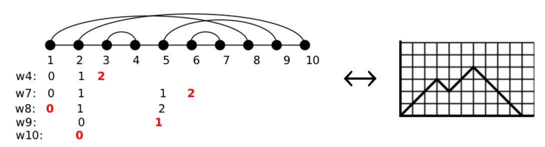
- B_{2n}: matchings on $\{1, 2, \ldots, 2n\}$
- $\mathcal{D}_{2n}^{<w_1>}$: Dyck paths of length n with weight vector $w_1 = (w_1, w_2, \ldots, w_{2n})$

Steps

Vertex type	Step
	╱
	╲

Weight may be up to $h_i - 1$, the maximum height at step i minus 1.

Example.



Weight vector $w1 = (0, 0, 0, 2, 0, 0, 2, 0, 1, 0)$.

Partitions

Bijection Φ_2

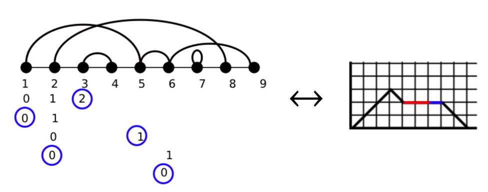
- $\mathcal{P}_n$: Partitions on the set $\{1, 2, \ldots, n\}$.
- $\mathcal{M}_n^{<w2>}$: Bicolored weighted Motzkin paths of length n with weight vector $w2 = (w_1, w_2, \ldots, w_n)$

Steps

Vertex type	Step
	╱
	╲
	—
	—

Weight may be up to $h_i - 1$, the maximum height at step i minus 1.

Example



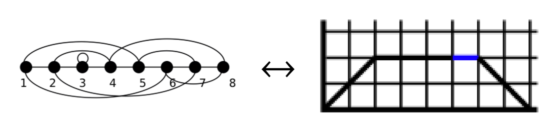
Weight vector $(0, 0, 0, 2, 0, 1, 0, 0, 0)$.

Permutations

Bijection Φ_3

- $\mathfrak{S}_n$: permutations of $\{1, 2, \ldots, n\}$.
- $\mathcal{M}_n^{<w3>}$: bicolored Motzkin paths of length n with weight vector $w3 = (w_1, w_2, \ldots, w_n)$. Each w_i may be up to the height of step i.

Example.



Weight vector: $(0, 0, 2, 1, 0, 0, [1, 0], [0, 0])$

Graphs

- May have multi-edges.
- A bijection with lattice paths is not obvious.

Continued fractions

We use the bijections to find a **single** generating function for three of the classes, marking length and nestings.

- Flajolet 1980 enumerates Motzkin paths using continued fractions.
- Let $M^{[i]}$ be the generating function for Motzkin paths of maximum height i.
- Mark north steps of height i by a_i, south steps of height i by b_i and east steps of height i by c_i.
- $M^{[0]} = \frac{1}{1-c_0}$, (a series of east steps.)
- $M^{[1]} = \frac{1}{1-c_0-\frac{a_0 b_1}{1-c_1}}$.
- $M^{[i]} = \frac{1}{1-c_0-\frac{a_0 b_1}{1-c_1-\frac{a_1 b_2}{\ddots \frac{}{1-ci}}}}$

In the bijections seen above, a weight greater than 0 in the lattice path indicated a nesting. The greatest weight that could be assigned was $h_i - 1$, the maximum height at that step minus one.

Let x mark length and y mark number of nestings. We perform the mapping:

$a_i \to x$

$b_i \to \frac{1}{2}q(q-1)(1-y^i)\frac{x}{(1-y)} - q(q-2)(1-y^i)\frac{x}{(1-y)} + \frac{1}{2}(q-1)(q-2)(1-y^i)^2\frac{x}{(1-y)^2}$

$c_i \to -q(q-2)\left(x + \frac{x(1-y^i)}{(1-y)}\right) + \frac{1}{2}(q-1)(q-2)\left((1-y^{(i+1)})\frac{x}{(1-y)} + \frac{(1-y^i)x}{(1-y)}\right)$

We can expand these expressions.

Matchings $q = 2$:

$$B(x,y) = 1+(1)x^2+(2+y)x^4+(5+6y+3y^2+y^3)x^6+(14+28y+28y^2+20y^3+\ldots$$

Partitions $q = 1$:

$$P(x,y) = 1+x+2x^2+5x^3+(y+14)x^4+\left(y^2+9y+42\right)x^5+(2y^3+14y^2+\ldots$$

Permutations $q = 0$:

$$S(x,y) = 1+(1)x+(2)x^2+(5+y)x^3+(14+8y+2y^2)x^4+(y^4+7y^3+25y^2+\ldots$$

From the diagram we see 1 matching with a nesting (blue), 1 partitions with a nesting (green), 8 permutations with one nesting (red) and 2 with two nestings (yellow).

New result

Theorem:[Burrill and Mishna] Define:

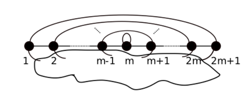
$NE(n,k) :=$ the number of permutations in S_n with a k-nestings.

$CR(n,k) :=$ the number of permutations in S_n with a k-crossing.

Then,

$$NE\left(n, \left\lceil\frac{n}{2}\right\rceil\right) = \begin{cases} m! & \text{if } n = 2m+1; \\ (m-1)!(2m^2-1) + 2(m!) - 1 & \text{if } n = 2n. \end{cases}$$

Proof: n=2m+1:



Conjecture For all $n > 2$, $k > 0$, $NE(n,k) = CR(n,k)$ where k is maximal.
Evidence: For $n < 10$:

$n\backslash k$	1	2	3	4	5	6
4	14	10				
5	42	76	2			
6	132	543	45			
7	429	3904	701	6		
8	1430	29034	9623	233		
9	4862	225753	126327	5914	24	