The Erdős - Turán Conjecture

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Introduction

A set $B \subset \mathbb{N}$ is called a basis of $\mathbb{N}$ if every natural number can be written as a sum of *two* elements of B. Define the *additive representation function* of B on $\mathbb{N}$ as

$$r(B,n) = \#\{(x,y) \in B^2 : x + y = n\}$$

Then a basis set B is one that satisfy $r(B,n) > 0$ for all $n \in \mathbb{N}$.
The Erdős and Turán [1] conjecture states:

Conjecture 1 *For any **basis** set B, $r(B,n)$ is unbounded. Equivalently, for any set $B \subset \mathbb{N}$,*

$$r(B,n) > 0, \forall n \in \mathbb{N} \longrightarrow \limsup_{n\to\infty} r(B,n) = \infty \qquad (1)$$

The conjecture asserts that if we attempt to *fill* the naturals with a set through pairwise addition, then the number of repeats in the additive representation will grow without bounds.
It will be more convenient to use the following equivalent form of (1):

$$r(B,n) \text{ is bounded} \longrightarrow r(B,n) = 0 \text{ for infinitely many values of } n. \qquad (2)$$

Note that we do not write there *exists* some values of n for which $r(B,n) = 0$, because if only finitely many n satisfy this, we can fill these holes without altering the boundedness of $r(B,n)$, thus producing a counter-example.

Finite constructions

To analyze the conjecture, we first model it in its finite form, and construct finite bases to test the validity of the conjecture.

We make the following definitions:

- We call a set $A = \{a_1, a_2, \ldots, a_m\}$, where $0 \le a_1 < a_2 < \ldots$, a finite basis if $r(A,n) > 0$ for all $n \le a_n$.
- A basis A is a k-basis if $r(A,n) \le k$ for all $n \in \mathbb{N}$.
- The set $E(k)$ contains all possible k-bases.

From these we gather 3 important observations:

(A) It is clear that $E(1) \subset E(2) \subset \ldots$

(B) Any finite truncation of a finite/infinite basis must be a finite k-basis, for some k.

(C) Hence, any infinite bases B belongs to

$$\Sigma = \lim_{k\to\infty} E(k)$$

The significance here is that, every k-basis has $r(A,n)$ bounded, and thus, if the conjecture is true, (2) implies that

there are no infinite k-bases for any k. $\qquad (*)$

The Algorithm

To compute the sets $E(k)$, we define $E_n(k)$ as the set of all bases in $E(k)$ that has exactly n elements. Now,

> Any set $A = \{a_1, a_2, \ldots, a_n\} \in E_n(k)$ is a *finite*, *k-basis* with n elements. Hence, by (B), its truncation $A \setminus \{a_n\}$ *must* be in $E_{n-1}(k)$.

So to compute $E(k)$, we first compute $E_1(k)$, then extend it do $E_2(k)$, etc.
In the end, $E(k) = E_1(k) \cup E_2(k) \cup \ldots$.

Note: The only finite basis with 1 element is $\{0\}$.

Given a set $A = \{a_1, a_2, \ldots, a_n\} \in E_n(k)$, we try to extend it by adding a_{n+1}.
- Recall that a finite k-basis satisfy $r(A,n) > 0$ for all $n \le a_n$.
- Clearly, $a_{n+1} > a_n$, for otherwise it would be discovered as a candidate earlier.
- Also, since $A' = A \cup \{a_{n+1}\}$ must also be a finite k-basis, we get:

> a_{n+1} must be $\le$ the first element n for which $r(A,n) = 0$.

The explanation is simple. Suppose $r(A,n) = 0$ and we choose $a_{n+1} > n$. Since a_{n+1} is the only new element, we can make $r(A',n) > 0$ only if there exist some a_i such that $a_i + a_{n+1} = n$. But this is impossible since $a_{n+1} > n$.

Thus, the largest possible candidate for n is $2a_n + 1$,
$\Longrightarrow$ When extending A, $\mathbf{a_n < a_{n+1} \le 2a_n + 1}$.

ET-COMPUTE(k)

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E(k) ← ∅
E_1(k) ← {{0}}
n ← 1
while E_n(k) ≠ {}
do E_{n+1}(k) ← ∅
    for each A ∈ E_n(k)
    do ▷ Here we take A = {a_1, a_2, ..., a_n}
        for a_{n+1} ← a_n + 1 to 2a_n + 1
        do A' ← A ∪ {a_{n+1}}
            if r(A', n) > 0 for all n ≤ a_{n+1} AND
               r(A', n) ≤ k for all n ≤ 2a_{n+1} + 1
               then E_{n+1}(k) ← E_{n+1}(k) ∪ A'
            if r(A, a_{n+1}) = 0
               then break for
    E(k) ← E(k) ∪ E_n(k)
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Analysis

If the conjecture is true, then this algorithm will *terminate* for each k.
i.e. $E_n(k) = \emptyset$ for some n.

While we do not know whether it will terminate for each k, we do know that the algorithm has complexity $O(MN)$ in the inner loop for each extension from $E_n(k)$ to $E_{n+1}(k)$, where M is the number of elements in $E_n(k)$, and N is the largest element in all of the sets in $E_n(k)$.

Results

The results for $E(3)$, $E(4)$, and $E(5)$ were computed easily. However, the size of $E(6)$ is evidently too large for a single computer to handle. Hence, to compute $E(6)$, we used the power of Apple's Xgrid.
Xgrid is a software that turns a cluster of Macs into a supercomputer. It provides parallel computation by queuing multiple jobs and distribute them to the cluster when there are free resources.

For $E(6)$, we first computed $E_6(6)$, which has 65 elements, and we submit a job to Xgrid for each of these 65 elements, using it as the starting point of the search. We then combine the results.

Results for $E(3)$:

Total number of bases:	9
Maximum non-empty level:	5
Maximum element in all bases:	8
Size of $E_1(3)$ =	1
Size of $E_2(3)$ =	1
Size of $E_3(3)$ =	2
Size of $E_4(3)$ =	3
Size of $E_5(3)$ =	2

Results for $E(4)$:

Total number of bases:	404
Maximum non-empty level:	12
Maximum element in all bases:	40
Size of $E_1(4)$ =	1
Size of $E_2(4)$ =	1
Size of $E_3(4)$ =	2
Size of $E_4(4)$ =	5
Size of $E_5(4)$ =	15
Size of $E_6(4)$ =	38
Size of $E_7(4)$ =	89
Size of $E_8(4)$ =	122
Size of $E_9(4)$ =	86
Size of $E_{10}(4)$ =	38
Size of $E_{11}(4)$ =	6
Size of $E_{12}(4)$ =	1

Results for $E(5)$:

Total number of bases:	6,335
Maximum non-empty level:	14
Maximum element in all bases:	52
Size of $E_1(5)$ =	1
Size of $E_2(5)$ =	1
Size of $E_3(5)$ =	2
Size of $E_4(5)$ =	5
Size of $E_5(5)$ =	17
Size of $E_6(5)$ =	60
Size of $E_7(5)$ =	201
Size of $E_8(5)$ =	552
Size of $E_9(5)$ =	1,100
Size of $E_{10}(5)$ =	1,568
Size of $E_{11}(5)$ =	1,580
Size of $E_{12}(5)$ =	937
Size of $E_{13}(5)$ =	285
Size of $E_{14}(5)$ =	46

Results for $E(6)$:

Total number of bases:	11,482,910,373
Maximum non-empty level:	35
Maximum element in all bases:	264
Size of $E_1(6)$ =	1
Size of $E_2(6)$ =	1
Size of $E_3(6)$ =	2
Size of $E_4(6)$ =	5
Size of $E_5(6)$ =	17
Size of $E_6(6)$ =	65
Size of $E_7(6)$ =	287
Size of $E_8(6)$ =	1,321
Size of $E_9(6)$ =	6,343
Size of $E_{10}(6)$ =	30,221
Size of $E_{11}(6)$ =	139,151
Size of $E_{12}(6)$ =	603,811
Size of $E_{13}(6)$ =	2,426,694
Size of $E_{14}(6)$ =	8,860,674
Size of $E_{15}(6)$ =	28,978,826
Size of $E_{16}(6)$ =	83,731,261
Size of $E_{17}(6)$ =	211,235,073
Size of $E_{18}(6)$ =	460,185,450
Size of $E_{19}(6)$ =	857,598,737
Size of $E_{20}(6)$ =	1,354,122,593
Size of $E_{21}(6)$ =	1,797,582,753
Size of $E_{22}(6)$ =	1,989,846,915
Size of $E_{23}(6)$ =	1,821,587,616
Size of $E_{24}(6)$ =	1,369,557,963
Size of $E_{25}(6)$ =	839,984,280
Size of $E_{26}(6)$ =	417,713,111
Size of $E_{27}(6)$ =	167,597,147
Size of $E_{28}(6)$ =	53,944,794
Size of $E_{29}(6)$ =	13,841,595
Size of $E_{30}(6)$ =	2,817,369
Size of $E_{31}(6)$ =	453,040
Size of $E_{32}(6)$ =	57,203
Size of $E_{33}(6)$ =	5,615
Size of $E_{34}(6)$ =	412
Size of $E_{35}(6)$ =	27

References

[1] P. Erdös and P. Turán. On a problem of Sidon in additive number theory, and on some related problems. *J. London Math. Soc.*, 16:212–215, 1941.