Visualizing Groups in Maple

Asif Zaman, Michael Monagan

aaz2@sfu.ca, mmonagan@cecm.sfu.ca

SIMON FRASER UNIVERSITY
THINKING OF THE WORLD

NSERC CRSNG

Overview

The structure and information of a group, G, can be represented in many ways. One can compute its entire multiplication table, also known as its Cayley Table, or determine all the subgroups of G and arrange them in a lattice, or identify isomorphic forms of the same group. We have implemented these new prospective additions for the *group* and the upcoming *FiniteGroups* package of Maple.

New representations for groups have also been added. Well known families of groups such as the Alternating Group or specific groups such as the Tetrahedral group can be called by name in *symbol* format. In addition, there is a matrix representation for groups in the form of *MatrixGroup*(p, S). The group is formed with a set of generators S under multiplication modulo p. If $p = 0$, then we are multiplying normally.

Cayley Table

Given a set of generators S for a group G, *DrawCayleyTable* draws the group's Cayley Table using Dimino's Algorithm to generate all the elements of G. Each element is associated with a distinct colour and label. There are various labelling, ordering, and colouring options for every element.

```
> G := permgroup(3, {[[1,2,3]], [[1,2]]}):
> DrawCayleyTable(G, labels = ’cycles’);
```

	()	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
()	()	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
(1 2)	(1 2)	()	(1 3 2)	(1 2 3)	(2 3)	(1 3)
(1 3)	(1 3)	(1 2 3)	()	(1 3 2)	(1 2)	(2 3)
(2 3)	(2 3)	(1 3 2)	(1 2 3)	()	(1 3)	(1 2)
(1 2 3)	(1 2 3)	(1 3)	(2 3)	(1 2)	(1 3 2)	()
(1 3 2)	(1 3 2)	(2 3)	(1 2)	(1 3)	()	(1 2 3)

S_3

With a subgroup H, one can display its coset partition in G. Elements in the same coset are coloured identically, and by default are grouped together. If H is normal in G, such as V_4 in S_4, then this makes for a particularly interesting display.

```
> G := permgroup(4, {[[1,2,3,4]], [[1,2]]}):
> H := permgroup(4, {[[1,2],[3,4]], [[1,3],[2,4]]}):
> DrawCayleyTable(G, coset = H);
```

	e	h	i	j	a	g	w	u	b	x	f	s	c	v	t	d	k	r	o	n	l	q	p	m
e	e	h	i	j	a	g	w	u	b	x	f	s	c	v	t	d	k	r	o	n	l	q	p	m
h	h	e	j	i	g	a	u	w	x	b	s	f	v	c	d	t	r	k	n	o	q	l	m	p
i	i	j	e	h	w	u	a	g	f	s	b	x	t	d	c	v	o	n	k	r	p	m	l	q
j	j	i	h	e	u	w	g	a	s	f	x	b	d	t	v	c	n	o	r	k	m	p	q	l
a	a	g	u	w	e	h	j	i	m	p	l	q	o	n	r	k	d	t	c	v	f	s	x	b
g	g	a	w	u	h	e	i	j	p	m	q	l	n	o	k	r	t	d	v	c	s	f	b	x
w	w	u	g	a	i	j	h	e	q	l	p	m	k	r	n	o	v	c	t	d	b	x	s	f
u	u	w	a	g	j	i	e	h	l	q	m	p	r	k	o	n	c	v	d	t	x	b	f	s
b	b	s	f	x	k	n	o	r	e	j	i	h	p	q	l	m	a	u	w	g	t	v	c	d
x	x	f	s	b	r	o	n	k	h	i	j	e	m	l	q	p	g	w	u	a	d	c	v	t
f	f	x	b	s	o	r	k	n	i	h	e	j	l	m	p	q	w	g	a	u	c	d	t	v
s	s	b	x	f	n	k	r	o	j	e	h	i	q	p	m	l	u	a	g	w	v	t	d	c
c	c	t	v	d	l	p	q	m	n	r	o	k	e	i	h	j	s	x	f	b	a	w	g	u
v	v	d	c	t	q	m	l	p	o	k	n	r	h	j	e	i	f	b	s	x	g	u	a	w
t	t	c	d	v	p	l	m	q	r	n	k	o	i	e	j	h	x	s	b	f	w	a	u	g
d	d	v	t	c	m	q	p	l	k	o	r	n	j	h	i	e	b	f	x	s	u	g	w	a
k	k	n	r	o	b	s	x	f	d	c	t	v	w	g	u	a	m	l	p	q	i	h	j	e
r	r	o	k	n	x	f	b	s	t	v	d	c	u	a	w	g	p	q	m	l	j	e	i	h
o	o	r	n	k	f	x	s	b	v	t	c	d	a	u	g	w	q	p	l	m	e	j	h	i
n	n	k	o	r	s	b	f	x	c	d	v	t	g	w	a	u	l	m	q	p	h	i	e	j
l	l	p	m	q	c	t	d	v	u	g	a	w	f	b	x	s	j	h	e	i	o	k	r	n
q	q	m	p	l	v	d	t	c	w	a	g	u	s	x	b	f	i	e	h	j	n	r	k	o
p	p	l	q	m	t	c	v	d	g	u	w	a	b	f	s	x	h	j	i	e	k	o	n	r
m	m	q	l	p	d	v	c	t	a	w	u	g	x	s	f	b	e	i	j	h	r	n	o	k

Recall $S_4/V_4 \cong S_3$!

One can display the conjugacy classes of the group.

```
> G := ‘Quaternions‘:
> DrawCayleyTable(G, conjugacy = true);
```

	1	−1	−j	j	−i	i	−k	k
1	1	−1	−j	j	−i	i	−k	k
−1	−1	1	j	−j	i	−i	k	−k
−j	−j	j	−1	1	−k	k	−i	−i
j	j	−j	1	−1	−k	k	i	−i
−i	−i	i	k	−k	−1	1	−j	j
i	i	−i	−k	k	1	−1	j	−j
−k	−k	k	−i	i	j	−j	−1	1
k	k	−k	i	−i	−j	j	1	−1

Classic representation of Q_8

With multiple subgroups $H_1, \cdots, H_n$, one can highlight the elements of each H_i in the Cayley Table. Elements in the same subgroup are coloured identically, but if an element, such as the identity, is in more than one subgroup, the element's colour is blended. Note that a subgroup can either be specified in a *group* format or as a set of generators.

```
> G := ‘Alt[4]‘:
> H1 := { [[1,2],[3,4]], [[1,3],[2,4]]}:
> H2 := permgroup(4, {[[1,2,3]]}):
> H3 := {[[2,3,4]]}:
> DrawCayleyTable(G, subgps = [H1, H2, H3]);
```

	e	a	b	c	d	g	k	l	f	h	i	j
e	e	a	b	c	d	g	k	l	f	h	i	j
a	a	e	c	b	l	j	f	d	k	i	h	g
b	b	c	e	a	i	k	g	h	j	l	d	f
c	c	b	a	e	h	f	j	i	g	d	l	k
d	d	h	l	i	g	e	a	f	b	k	j	c
g	g	k	f	j	e	d	h	b	l	a	c	i
k	k	g	j	f	b	i	l	e	h	c	a	d
l	l	i	d	h	j	a	e	k	c	f	g	b
f	f	j	g	k	c	h	d	a	i	b	e	l
h	h	d	i	l	f	c	b	g	a	j	k	e
i	i	l	h	d	k	b	c	j	e	g	f	a
j	j	f	k	g	a	l	i	c	d	e	b	h

1
2
3

Isomorphism Test

The isomorphism problem is an exponentially hard problem. Hence, finding the best invariants to distinguish two groups is crucial. Our computational experiments strongly suggest that the conjugacy classes and commutativity relations in a group are good tests for isomorphism. Using these invariants (and a few others), the *IsIsomorphic* procedure can determine whether two groups G_1, G_2 are isomorphic.

If $G_1 \ncong G_2$, the procedure returns *false*, along with an explanatory message.

```
> A := Matrix([[0,I],[I,0]]):
> B := Matrix([[0,-1],[1,0]]):
> G1 := MatrixGroup(0, {A, B}):  # Quaternions
> G2 := ‘Dihedral[8]‘:
> IsIsomorphic(G1, G2);
```

false, "Element Order failure"

If $G_1 \cong G_2$, the procedure returns *true*, along with an isomorphic map from G_1 to G_2.

```
> G1 := permgroup(8, {[[1,2,3,4],[5,6,8,7]], [[1,5,3,8], [2,7,4,6]]}):
> G2 := ‘Quaternions‘:
> b, phi := IsIsomorphic(G1, G2);
```

true, Φ

The given map can take any element from G_1 and returns its corresponding element in G_2. Note that the map returned is not necessarily unique.

```
> phi([]);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
> phi([[1,2,3,4],[5,6,8,7]]);
```

$$\begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}$$

```
> phi([[1,3],[2,4],[5,8],[6,7]]);
```

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

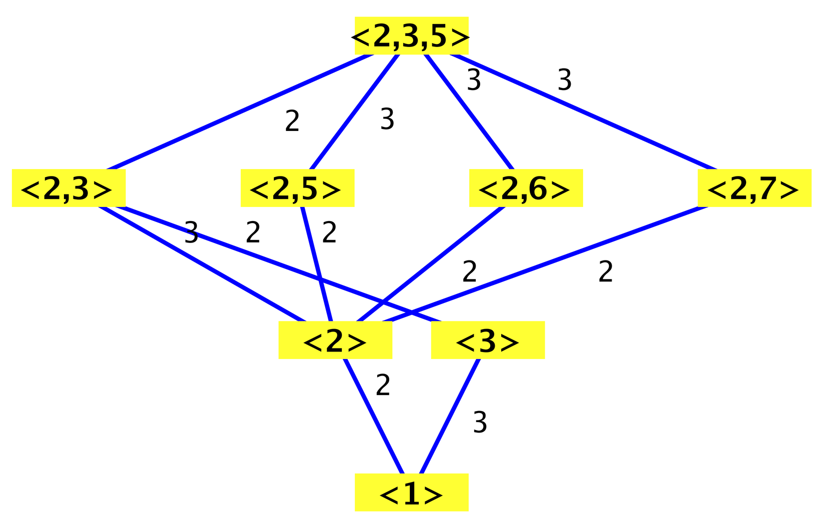
Subgroup Lattice

If G is a group, then its **subgroup lattice**, $L(G) = (V, E)$, is defined as follows:

- ⋄ The vertex set V is composed of all distinct subgroups H in G.
- ⋄ If $|H|$ is a product of exactly i primes, then H is in level i of $L(G)$, denoted $H \in L^{(i)}(G)$.
- ⋄ Suppose $H \in L^{(i)}(G)$ and $K \in L^{(j)}(G)$ where $j > i$. Then $\{H, K\} \in E$ if and only if $H \subset K$ and $j - i$ is minimal.
- ⋄ If $\{H, K\} \in E$ and $|H| < |K|$, then its edge weight is defined as the index between the two subgroups $|K : H| = |K|/|H|$.

Given a solvable group G, the procedure *DrawSubgroupLattice* can display the complete $L(G)$ using the Cyclic Extension Method. *DrawSubgroupLattice* returns the plot or graph of $L(G)$, and the list of elements of G. Each vertex is labelled with the generators of its group; the generators are labelled by the corresponding index in the returned list of elements, below defined as S.

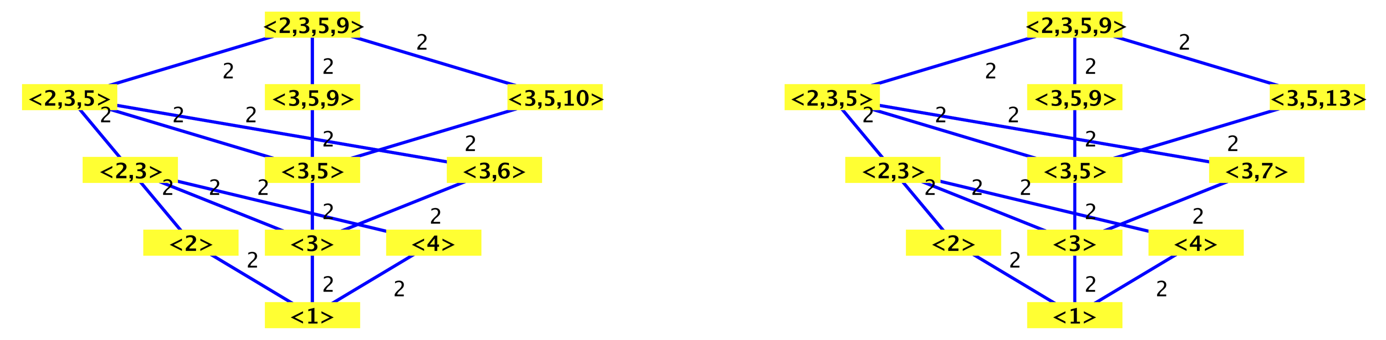
```
> G := grelgroup({s,t}, {[s,s,s,s,s,s],[s,s,s,1/t,1/t],[s,t,s,1/t]}):
> P, S := DrawSubgroupLattice(G, output = ‘plot‘):
> P;
```



Although the subgroup lattice contains a wealth of information about a group, it is not unique. The smallest such example exists at order 16. Below we have $G_1 \cong C_8 \times C_2$ and $G_2 =< x, y | x^8 = y^2 = y^{-1}x^{-1}yx^5 = e >$. Clearly, $G_1 \ncong G_2$, yet $L(G_1) \cong L(G_2)$.

```
> G1 := permgroup(10, {[[1,2,3,4,5,6,7,8]], [[9,10]]}):
> G2 := grelgroup({x,y}, {[seq(x,i=1..8)],[y,y],[1/y,1/x,y,x,x,x,x,x]}):
> IsIsomorphic(G1,G2);
```

false, "Commutativity failure"



G_1 G_2

If G is not solvable, then G contains perfect subgroups. Unfortunately, the Cyclic Extension Method cannot generate the perfect subgroups of a group, so $L(G)$ is currently incomplete for non-solvable groups.

Future Projects

- ⋄ Inclusion of all perfect subgroups in *DrawSubgroupLattice*
- ⋄ Draw the Cayley Graph given a set of generators for a group
- ⋄ Efficient group representation for Matrices over a Galois Field
- ⋄ Highlighting of special subgroups in $L(G)$, such as normal ones.

Special thanks to Greg Fee, Kseniya Garaschuk, and Paul Vrbik for their help and advice.